Vibrações e Ondas 2015 - Lista 2 - 22/04/2015

Exercícios: 7, 10, 11, 13, 15, 16 : French cap. 4

⑦ $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \Rightarrow x_p(t) = \frac{F_0}{m}\frac{\cos(\omega t + \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$

(a) Energia cinética instantânea?

$$E_c = \frac{1}{2} m \dot{x}^2 \quad , \quad \dot{x}(t) = -\frac{F_0}{m}\omega \frac{\sin(\omega t + \varphi)}{[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2]^{1/2}}$$

$$\therefore E_c = \frac{1}{2} m \left(\frac{F_0 \omega}{m}\right)^2 \frac{\sin^2(\omega t + \varphi)}{[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2]} \,//$$

(b) Energia potencial instantânea? $\quad \omega_0^2 = K/m$

$$E_p = \frac{1}{2} K x^2 = \frac{1}{2} m \omega_0^2 \left(\frac{F_0}{m}\right)^2 \frac{\cos^2(\omega t + \varphi)}{[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2]}$$

(c) $$\langle f \rangle = \frac{1}{T}\int_0^T f(t)dt$$

$$\langle E_c \rangle = \frac{1}{T}\int_0^T \frac{1}{2} m \left(\frac{F_0 \omega}{m}\right)^2 A^2 \sin^2(\omega t + \varphi)\, dt$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\langle E_c \rangle = \frac{1}{T}\int_0^T \frac{1}{2} m \frac{F_0^2 \omega^2}{m^2} A^2(\omega) \left(\frac{1}{2} - \frac{\cos(2\omega t + 2\varphi)}{2}\right) dt$$

$$\langle E_c \rangle = \frac{1}{T}\ \frac{1}{2} m \left(\frac{F_o\,\omega}{m}\right)^2 A^2(\omega) \int_0^T \left(\frac{1}{2} - \frac{\cos(2\omega t + 2\varphi)}{2} \right) dt$$

$$= (\ldots)\left[\frac{1}{2} - \frac{1}{2}\int_{2\varphi}^{2\omega T + 2\varphi} \cos u \ \frac{du}{2\omega} \right] = (\ldots)\left[\frac{1}{2} - \frac{1}{2}\frac{\sin(2\omega t + 2\varphi)}{(2\omega)} \Big|_0^T \right]$$

$$= (\ldots)\left[\frac{1}{2}T - \frac{1}{4\omega}\left(\sin(2\omega T + 2\varphi) - \sin(2\varphi) \right) \right]$$

$$= (\ldots)\left[\frac{1}{2}T - \frac{1}{4\omega} \underbrace{\left(\sin\left(2\cancel{\omega}\cdot\frac{2\pi}{\cancel{\omega}} + 2\varphi \right) - \sin(2\varphi) \right)} \right] = (\ldots)\ \frac{1}{2}T$$

$$= \sin(4\pi + 2\varphi) - \sin(2\varphi)$$

$$= \cancel{\sin(4\pi)}^{\,0} \cos(2\varphi) + \sin(2\varphi)\cancel{\cos(4\pi)}^{\,1} - \sin(2\varphi) = 0$$

$$\therefore \langle E_c \rangle = \frac{1}{\cancel{T}}\ \frac{1}{2}\cancel{m}\ \frac{F_o^2\,\omega^2}{m^{\cancel{2}}}\ A^2(\omega)\cdot \frac{1}{2}\cancel{T} \Rightarrow \boxed{\langle E_c \rangle = \frac{1}{4}\frac{F_o^2\,\omega^2}{m} A^2(\omega)}$$

$$\langle E_p \rangle = \frac{1}{2}\cancel{m}\,\omega_o^2\ \frac{F_o^2}{m^{\cancel{2}}}\cdot A^2\ \underbrace{\langle \cos^2(\omega t + \varphi) \rangle}_{1/2}$$

$$\boxed{\langle E_p \rangle = \frac{1}{4}\frac{F_o^2\,\omega_o^2 A^2(\omega)}{m}}$$

$$\therefore \boxed{\frac{\langle E_c \rangle}{\langle E_p \rangle} = \left(\frac{\omega}{\omega_o}\right)^2}$$

(d) para $\omega = \omega_o$: $\langle E_c \rangle = \langle E_p \rangle$

$$E_T(t) = E_c(t) + E_p(t)$$

$$E_T(t) = \frac{1}{2}\frac{F_o^2\,\omega^2}{m} A^2 \sin^2(\omega t + \varphi)$$

$$+ \frac{1}{2}\omega_o^2\ \frac{F_o^2}{m} A^2 \cos^2(\omega t + \varphi)$$

$$\therefore \boxed{E_T(t,\ \omega = \omega_o) = \frac{1}{2}\frac{F_o^2\,\omega_o^2}{m} A^2(\omega)}$$

$$E_T(t) = \frac{1}{2}\frac{F_0^2\omega^2}{m}A^2\sin^2(\omega t+\varphi) + \frac{1}{2}\frac{F_0^2\omega_0^2}{m}A^2\cos^2(\omega t+\varphi)$$

$$\frac{d}{dt}E_T(t) = \frac{1}{2}\frac{F_0^2\omega^2A^2}{m}\left[2\omega\sin(\omega t+\varphi)\cos(\omega t+\varphi)\right] + \frac{1}{2}\frac{F_0^2\omega_0^2}{m}A^2\left[2\omega\sin(\omega t+\varphi)\cos(\omega t+\varphi)\right]$$

$$\frac{d}{dt}E_T(t) = \frac{1}{2}\frac{F_0 A^2}{m}2\omega\sin(\omega t+\varphi)\cos(\omega t+\varphi)\left(\omega^2-\omega_0^2\right)$$

Para $E_T(t)$ ser constante no tempo: $\frac{d}{dt}E_T(t) = 0$

$$\therefore \frac{d}{dt}E_T(t) = 0 \Rightarrow \boxed{\omega = \omega_0}$$

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(a) $$W = -\int_C \vec{F}\cdot d\vec{r} = -\int_{x_0}^{x_1} -b\,v(t)\,dx = \int_{t_0}^{t_1} b\,v^2(t)\,dt$$

$$\left\langle \frac{dx}{dt} = v(t)\right.$$

$$\therefore \boxed{\frac{dW}{dt} = b\,v^2(t)}$$

(b) $x = A\cos(\omega t-\varphi)$, $\dot{x} = -A\omega\sin(\omega t-\varphi)$

$$\langle P\rangle = \left\langle\frac{dW}{dt}\right\rangle = \frac{1}{T}\int_0^T b\,A^2\omega^2\sin^2(\omega t-\varphi)\,dt \Rightarrow \boxed{\langle P\rangle = \frac{b\omega^2A^2}{2}}$$

(c) $$A^2 = \frac{F_0^2/m^2}{(\omega_0^2-\omega^2)^2+(2\gamma\omega)^2}, \quad Q = \frac{\omega_0}{2\gamma}, \quad 2\gamma = b/m$$

$$\therefore \langle P\rangle = \frac{b\omega^2}{2}\frac{F_0^2/m^2}{\left[(\omega_0^2-\omega^2)^2+(2\gamma\omega)^2\right]}$$

$\omega_0^2 = K/m \ , \ 2\gamma = b/m \ , \ Q = \omega_0/2\gamma$

$$\langle P \rangle = \frac{\cancel{\omega}^{\cancel{2}}}{\cancel{2}} (\cancel{2}\gamma) \cdot \frac{F_0^2/m}{\omega_0^2 \cancel{\omega^2}\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]} = \frac{F_0^2}{\cancel{m}} \cdot \frac{\gamma}{\frac{K}{\cancel{m}}} \frac{1}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}$$

$$\langle P \rangle = \frac{F_0^2}{K} \cdot \frac{\omega_0}{2Q} \frac{1}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}$$

$$\boxed{\langle P \rangle = \frac{F_0^2 \omega_0}{2KQ} \frac{1}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}}$$

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$m = 0{,}2 \ Kg$ $\qquad F_0 = 2N$ $\qquad \omega_0^2 = \frac{K}{m}$

$b = 4 \ \frac{Ns}{m}$ $\qquad \omega = 30 s^{-1}$

$K = 80 \ N/m$ $\qquad F = F_0 \cos(\omega t)$ $\qquad 2\gamma = \frac{b}{m}$

(a)

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

$$x(t) = A(\omega) \cos(\omega t + \varphi) \ , \quad A(\omega) = \frac{F_0}{m} \frac{1}{[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2]^{1/2}}$$

$$\tan\varphi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

$$\therefore A(\omega = 30) = \frac{2}{0{,}2} \cdot \frac{1}{\left\{\left(\frac{80}{0{,}2} - 900\right)^2 + \left(\frac{4}{0{,}2} \cdot 30\right)^2\right\}^{1/2}} \Rightarrow \boxed{A(30) = 0{,}0128 \, m}$$

$$\varphi = \tan^{-1}\left(\frac{\frac{4}{0{,}2} \cdot 30}{-500}\right) = -50^\circ \text{ ou } 130^\circ$$

(b)

$$W = -\int b v\, dx = -\int b v^2 dt \quad , \quad v = -A\omega \sin(\omega t - \delta)$$

$$W = -\int_0^T b.\, A^2\omega^2 \sin^2(\omega t - \delta)\, dt = -bA^2\omega^2 \int_0^T \sin^2(\omega t - \delta)\, dt$$

$$W = -bA^2\omega^2 \left[\frac{1}{2}T - \frac{1}{4\omega} \left[\sin\left(2\omega . \frac{2\pi}{\omega} - 2\delta\right) + \sin(2\delta) \right] \right]$$

$$W = -\frac{bA^2\omega^2}{2} T \Rightarrow W = -\frac{bA^2}{\cancel{2}}\omega^{\cancel{2}} . \frac{\cancel{2}\pi}{\cancel{\omega}} \Rightarrow \boxed{W = -\pi b A^2 \omega}$$

$$\boxed{W = 0{,}062\ J}$$

(c)

$$P(t) = \frac{dW}{dt} = F_{ext}\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t).\left[-A\omega \sin(\omega t - \delta)\right]$$

$$\langle P \rangle = -\frac{1}{T}\int_0^T \frac{F_0}{m} A\omega \cos(\omega t)\sin(\omega t - \delta) = -\frac{F_0 A\omega}{m\, 4\pi}\left[-\sin\delta\,(2\pi)\right]$$

$$\therefore \boxed{\langle P \rangle = \frac{F_0 A\omega}{2} \sin\delta = 0{,}29\ W}$$

(13) $$\frac{d}{dt}\langle P \rangle = 0 \Rightarrow \omega = \omega_0 \Rightarrow \langle P_{max} \rangle = \frac{1}{2}\frac{F_0^2 \omega_0 Q}{K}$$

(a) $\omega_0 = 40\ s^{-1}$

$2\gamma = 2\ s^{-1} \Rightarrow \boxed{Q = 20}$

(b) $x(t) = Ae^{-\gamma t}\cos(\omega t+\varphi)$

$\dot{x}(t) = -\gamma A e^{-\gamma t}\cos(\omega t+\varphi) + Ae^{-\gamma t}\omega\sin(\omega t+\varphi)$

$E_T(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2e^{-2\gamma t}\left[\cos(\omega t+\varphi)\gamma - \omega\sin(\omega t+\varphi)\right]^2$

$+\frac{1}{2}\omega_0^2 mA^2 e^{-2\gamma t}\omega^2\sin^2(\omega t+\varphi)$, $T = \frac{2\pi}{\omega_0}$

$\frac{E_T(t=0)}{E(nT)} = e^5 = \frac{\frac{1}{2}mA^2(\gamma\cos\varphi - \omega\sin\varphi)^2 - \frac{1}{2}m\omega_0^2\omega^2A^2\sin\varphi}{(\ldots)}$

$e^{5-2\gamma\pi n/\omega} = 1 \Rightarrow 5 - \frac{2\pi n}{\omega}\gamma = 0$

como $\omega \approx \omega_0$: $5 = \frac{2\pi n}{\omega_0}\gamma \Rightarrow n = \frac{5\omega_0}{2\pi\gamma}$

$\boxed{n = 16 \text{ ciclos}}$

$\omega_1 \to \omega$

$\omega_0 \to \omega_1$ ω

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(a) $\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}$

metade pot $\Rightarrow 2\gamma = \omega_1/5$

pot max $= \omega_0$

$\omega_1^2 = \omega_0^2 - \frac{\omega_1^2}{10}$?

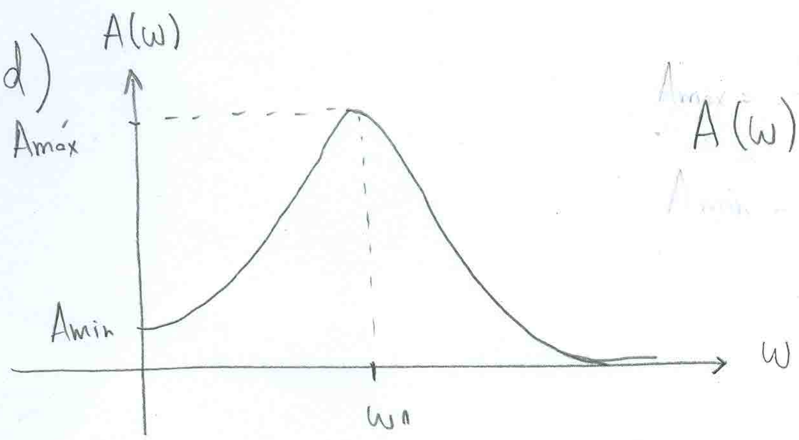
$\boxed{\omega_0 = \omega_1\frac{\sqrt{101}}{10}}$

(b) $Q = \frac{\omega_0}{2\gamma} = \frac{\omega_1 \sqrt{101}/10}{\omega_1/5} \Rightarrow \boxed{Q = \frac{\sqrt{101}}{2}}$

(c) $2\gamma = \frac{b}{m} \Rightarrow b = 2\gamma m = \frac{\omega_1}{5} m$

$b = \frac{m}{5} \cdot \frac{10}{\sqrt{101}} \omega_0 = \frac{m}{\cancel{5}} \cdot \frac{\overset{2}{\cancel{10}}}{\sqrt{101}} \sqrt{\frac{K}{m}} \Rightarrow \boxed{b = \frac{2\sqrt{km}}{\sqrt{101}}}$

(d)



$A(\omega) = \frac{F_0}{m} \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2\right]^{1/2}}$

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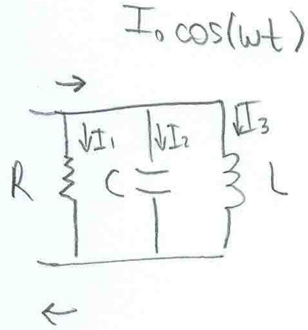

$V_R = I_1 R \quad (1)$

$V_c = L \frac{q_2}{C} \quad (2)$

$V_L = L \frac{dI_3}{dt} \quad (3$

$I = I_1 + I_2 + I_3$

$I = I_0 \cos(\omega t)$

(1) e (2) : $I_1 R = L \frac{dI_3}{dt} \Rightarrow I_1 = \frac{L}{R} \frac{dI_3}{dt}$

(2) e (3) : $\frac{q_2}{C} = L \frac{dI_3}{dt} \Rightarrow q_2 = LC \frac{dI_3}{dt} \Rightarrow I_2 = \frac{dq_2}{dt}$

$I_1 + I_2 + I_3 = I_0 \cos(\omega t)$

$\frac{L}{R} \frac{dI_3}{dt} + LC \frac{d^2 I_3}{dt^2} + I_3 = I_0 \cos(\omega t)$ $\begin{cases} \omega_0 = \sqrt{1/CL} \\ 2\gamma = 1/RC \end{cases}$

$$\langle P_{max} \rangle = \langle P(\omega_0) \rangle$$

$$\langle P_{max} \rangle = \frac{1}{2} \frac{F_0^2 \omega_0 Q}{K} = \frac{F_0^2 Q}{2m\omega_0} = \frac{F_0^2 \cancel{\omega_0}\, \omega_0}{2m \cancel{\omega_0}\, 2\gamma} = \frac{F_0^2}{4m\gamma}$$

$$\therefore \langle P_{max} \rangle = \frac{1}{2} I_0^2 \cdot \frac{\omega_0^2}{2\gamma} \Rightarrow \langle P_{max} \rangle = \frac{I_0^2}{2} \cdot \frac{R}{L}$$